MODES OF STABILITY LOSS IN AN ELASTIC ROD UNDER IMPACT

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If an elastic system is acted on by a shock load considerably in excess of the first critical value, the rate of increase in the bending is largest in a form differing from the first [1]. The character of the motion is substantially affected by the wave set up by the sudden application of the load [2, 3].

We can avoid assuming equal orders of smallness for the perturbations corresponding to various forms by asymptotic consideration of the deflection of an elastic hinged rod with an initial irregularity; we use expansion with respect to the inherent forms of loss of stability over a variable interval [4]. A simplified method is given below for deducing the curved form of the rod under impact as a system with one degree of freedom. This form is very similar to that found in tests on shock loading of rods [3]. An elastic three-layer rod is envisaged, and also a uniform ring with specific conditions of loading.

1. The following is [5] the system of equations that takes account of interaction of longitudinal and transverse vibrations in a uniform elastic rod with an initial irregularity:

$$EIw_{,xxxx} + EF(u_{,x}w_{,x})_{,x} + \rho Fw_{,tt} = f(x,t), \qquad (1.1)$$

$$u_{,xx} = \rho E^{-1} u_{,tt}. \qquad (1.2)$$

Here w and u are the normal and longitudinal displacements of a cross-section, x is the longitudinal coordinate, t is time, E is Young's modulus, I is bending rigidity, F is the cross-sectional area (both constant along the rod), ρ is density, and $f(x,t)$ is a function defined by the initial perturbations or imperfections.

The hinged rod has a length l_0 ($0 \le x \le l_0$). At one end is a massless support to which the shock load N_0 is applied at $t = 0$ (this load greatly exceeds the Euler critical load P_e^0). We need not consider the conditions of attachment at the other end if we consider times such that

$$t \leqslant t_0, \quad t_0 = l_0 / c, \quad c = (E/\rho)^{1/2},$$

in which c is the speed of sound in the material.

Equation (1.2) defines the load N acting along the rod:

$$N = N_0 \quad (x < ct), \quad N = 0 \quad (x \geqslant ct) \qquad (1.3)$$

Equation (1.1) takes the form

$$EIw_{,xxxx} + Nw_{,xx} + \rho Fw_{,tt} = f(x,t) \quad (0 \leqslant x \leqslant l_0). \qquad (1.4)$$

The bending perturbations defined by (1.1) are unimportant for $x > ct$, and we neglect them. We also assume that

$$f(x, t) \equiv 0 \quad (x > ct).$$

The boundary conditions for (1.4) are as follows for a variable interval [4]:

$$w = w_{,xx} = 0 \; (x = 0), \qquad w = w_{,x} = 0 \quad (x = l = ct).$$

The initial conditions for that equation are zero:

$$w = w_{,t} = 0 \quad (t = 0).$$

It is readily shown that the asymptotes to the natural forms in loss of stability are as follows ($m \to \infty$):

$$W_m(x) = \sin(m\pi x / l) \quad (x < l), \qquad W_m(x) = 0 \quad (x \geqslant l). \quad (1.5)$$

These are the natural forms of loss of stability of a hinged rod:

$$w = w_{,xx} = 0 \quad (x = 0, \; x = l).$$

Consider the asymptotic solution to (1.4)

$$w(x, t) = \sum_{m=1}^{\infty} q_m(t) W_m(x).$$

We use the condition that the $W_m(x)$ are orthogonal in a variable interval to get for the $q_m(t)$ ordinary differential equations with variable coefficients (minor terms are omitted):

$$\rho F q_m'' + (\pi / l)^4 EIm^2 \, (m^2 - \eta^2) q_m = f_m(t)$$

$$(m = 1, 2, \ldots). \qquad (1.6)$$

Here

$$l = ct, \qquad \eta^2 = \frac{N}{P_e}, \qquad P_e = \frac{\pi^2 EI}{l^2},$$

$$f_m(t) = \frac{2}{l} \int_0^l f(x, t) \sin \frac{m\pi x}{l} \, dx$$

for $t = t_0$

$$\eta_0^2 = \frac{N_0}{P_e^0}, \qquad P_e^0 = \frac{\pi^2 EI}{l_0^2}, \quad f_m^0 = \frac{2}{l_0} \int_0^{l_0} f(x) \sin \frac{m\pi x}{l_0} \, dx$$

Let some one term f_i^0 in the expansion be different from zero when $i < \eta_0$. Then $f_m(t) = 0$ if ct/m is a multiple of l_0/i, while $f_m(t) \ne 0$ at any other instant [see (1.6)], and then for all m at $t = t_0$ we have $q_m(t_0) \ne 0$, $q'_m(t_0) \ne 0$. Then, if we neglect reflection from the support $x = l_0$ for $t > t_0$, "the forms that alter most rapidly are the ones that correspond to the largest coefficient in the exponent in the time function relating to the corresponding motion" [1]. This approach makes it unnecessary to suppose equal orders of smallness for the initial perturbations that correspond to different static forms of loss of stability.

Consider (1.6). The coefficient of the second term may be positive, negative, or zero in accordance with N_0 and the number of the equation; in the first case we have vibration, while in the other two we have loss of stability. This coefficient has a maximum for loss of stability when

$$l^* = l/m = \sqrt{2} l_0 / \eta_0.$$

Also, l^* = const for a given rod with a definite compressive load, and this wavelength is independent of the number of the equation in (1.6).

We transform the coordinates for (1.4):

$$x = x, \quad \tau = t - x/c.$$

The spatial coordinate is unaltered, while the time coordinate is transformed into the true time of action of the compressive load at a given point along the rod. Then (1.4) becomes

$$EI[(\;)_{,x} - c^{-1} (\;)_{,\tau}]^4 w +$$

$$+ N[(\;)_{,x} - c^{-1} (\;)_{,\tau}]^2 w + \rho F w_{,\tau\tau} = f(x, \tau). \qquad (1.7)$$

It is shown below that the second terms in the brackets can be neglected relative to the first. We omit these terms, and the simplified (1.7) then has the form of (1.4). The maximum rate of increase in deflection occurs for wavelength l^*, so we construct the asymptotic solution to the simplified (1.7) as for a system with one degree of freedom:

$$w(x, \tau) = Q(\tau) W^*(x), \qquad W^*(x) = \sin \frac{\pi x}{l^*} \quad (0 \leqslant x \leqslant l),$$

$$W^*(x) = 0 \quad (x > l). \qquad (1.8)$$

After substitution, (1.8) gives for $Q(\tau)$ an ordinary differential equation with constant coefficients, with zero initial conditions:

$$Q''(\tau) - n^2 Q(\tau) = a^*, \quad n^2 = EI(\rho F)^{-1}(\pi / l^*)^4, \qquad (1.9)$$

if we assume that the right side of the equation has the form

$$f(x, \tau) = a^* \rho F \sin(\pi x / l^*) \ (0 \leqslant x \leqslant \alpha_0 = \text{const}).$$

The solution to (1.9) is

$$Q(\tau) = (a^* / n^2)(\operatorname{ch} n\tau - 1).$$

We revert to the old coordinates. The following is the asymptotic solution to (1.4) as a system with one degree of freedom for a time t_1 ($\alpha_0/c \leq t_1 \leq t_0$):

$$w(x, t_1) = \frac{a^*}{n^2}\left[\operatorname{ch}\left(nt_1 - \frac{nx}{c}\right) - 1\right]\sin\frac{\pi x}{l^*} \quad (0 \leqslant x < ct_1),$$

$$w(x, t_1) = 0 \qquad (x \geqslant ct_1). \qquad (1.10)$$

The deflection occurs most rapidly at a particular wavelength, and adjacent waves grow independently, each corresponding to its own mean time of loading (mean true time), while the maxima (minima) decrease exponentially away from the end to which the load is applied.

Note that if $c = c(x)$ (variable speed), the coordinate transformation is taken as

$$x = x, \qquad \tau = t - \int_0^x \frac{d\xi}{c(\xi)}.$$

The other arguments remain in force.

Consider the errors from discarding the small terms in (1.7). The second term in the brackets is small relative to the main one if

$$r / l^* << 1.$$

Here r is the radius of inertia of the cross-section. This condition may be given the form $N/F \ll E$, i.e., the wavelength for loss of stability must be much greater than the radius of inertia, while the stress must be much less than the elastic modulus, which are conditions for (1.1) and (1.2) to apply.

The solution of (1.10) agrees well with experiment [3]. The behavior of a long rod in shock compression was examined by high-speed cinematography.

If $c \to \infty$ in (1.10), the rod may be considered as a system with one degree of freedom, this degree corresponding to the maximum index in the exponent [1].

2. The following is the system of equations analogous to (1.1) and (1.2) for a three-layer rod with an initial irregularity [6]:

$$EI(\chi - \vartheta h^2 \beta^{-1} \chi_{,xx})_{,xxxx} +$$

$$+ EF(u_{,x} w_{,x})_{,x} + \rho F w_{,tt} = f(x, t) \qquad (2.1)$$

$$u_{,xx} = \rho E^{-1} u_{,tt} \qquad (2.2)$$

The normal displacement w of the cross-section is defined via the function χ:

$$w = \chi - h^2 \beta^{-1} \chi_{,xx}.$$

Here E is the reduced Young's modulus for the entire cross-section, I is as previously, h is the total thickness, and ϑ and β are coefficients for the bending rigidity of the facing layers and the shear flexibility of the filling, with $0 \leq \vartheta \leq 1$. Also, ρ is the reduced density per unit length of rod. The other symbols are as before. We envisage a rod for which $h/l_0 \ll 1$.

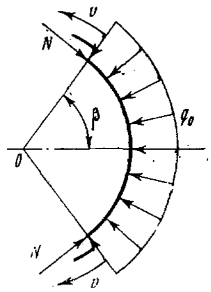
A load $N_0 \gg P_e^\circ$ is applied at $t = 0$ to the hinged end. The load N acting along the rod is defined by (1.3), with $c = E/\rho)^{1/2}$ perhaps much less than in a homogeneous material on account of the altered ρ and E.

Equation (2.1) becomes

$$EI(\chi - \vartheta h^2 \beta^{-1} \chi_{,xx})_{,xxxx} +$$

$$+ N(\chi - h^2\beta^{-1}\chi_{,xx})_{,xx} + \rho F(\chi - h^2\beta^{-1}\chi_{,xx})_{,tt} = f(x,t)$$

$$(0 \leqslant x \leqslant l_0). \qquad (2.3)$$



The initial conditions for (2.3) are zero, while the following are the boundary conditions for a variable interval:

$$\chi = \chi_{,xx} = \chi_{,xxxx} = 0 \quad (x = 0)$$

$$\chi_{,x} = \chi_{,xxx} = \chi - h^2\beta^{-1}\chi_{,xx} = 0 \quad (x = l = ct).$$

We still have (1.5) for the asymptote of the natural forms of loss of stability. Equation (2.3) simplifies considerably for large m:

$$(\vartheta EI\chi_{,xxxx} + N\chi_{,xx} + \rho F\chi_{,tt})_{,xx} = -h^{-2}\beta f(x, t).$$

We obtain the case considered above, thus, such a rod behaves like a homogeneous one having ϑEI as its bending rigidity.

3. Consider the buckling of a ring under a symmetrical shock load q applied at t = 0 to the ring at rest (the loading front propagates with a velocity v exceeding the speed of sound in the material, $c = (E/\rho)^{1/2}$:

$$q(\beta, t) = q_0 \ (-vt / R \leqslant \beta \leqslant vt / R),$$

$$q(\beta, t) = 0 \ (-\pi \leqslant \beta < -vt / R, \ vt / R < \beta \leqslant \pi).$$

We assume that load q deforms the ring without inertia (Fig. 1). Then the equation for the motion of a ring having an initial irregularity is

$$EIR^{-4} w_{,\beta\beta\beta\beta} + NR^{-2} w_{,\beta\beta} + \rho F w_{,tt} = f(\beta),$$

$$(N = qR). \qquad (3.1)$$

The load is a shock one, so $N_0 = q_0 R \gg P_e^\circ$. Equation (3.1) is considered in a variable interval, see section 1. The system with an infinite number of degrees of freedom is replaced by a system with one degree of freedom, but over a variable interval.

At time t_1 ($0 \leq t_1 \leq t_0$, $t_0 = \pi R/v$), the following is the asymptotic representation of the shape of the ring as a system with one degree of freedom:

$$w(\beta, t_1) = \frac{a^*}{n^2}\left[\operatorname{ch}\left(nt_1 - \frac{nl_0\beta}{\pi v}\right) - 1\right]\cos\frac{\beta l_0}{l^*}$$

$$\left(0 \leqslant \beta < \frac{vt}{R}\right),$$

$$w(\beta, t_1) = 0 \qquad (\beta \geqslant vt_1 / R). \qquad (3.2)$$

As $\pi x = l_0 \beta$, this formula differs from (1.10) only in that $\cos(\beta l_0/l^*)$ replaces $\sin(\pi x/l^*)$, which is due to the symmetry of the loading. The passage to the limit $v \to \infty$ in (3.2) still applies.

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